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COALESCING AND NONCOALESCING STOCHASTIC FLOWS IN R_1

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We study homogeneous stochastic flows, families X_{st}, $0 \leq s \leq t < \infty$ of random mappings of R_1 into itself, with the composition property $X_{tu} \circ X_{st} = X_{su}$, $s \leq t \leq u$, and with independent 'increments'. Depending on the di: rentiability at 0 of the covariance function of the field of small displacements, the mapping X_{st}, s and t fixed, may be smooth or may be a step function mapping R_1 into a countable set of points with no limit points. (The latter kind of situation has occurred in related work of R. Arratia.) It is not known whether other behavior is possible. Some results in the countable-range case are deduced from duality results obtained for the smooth case. The case of double exponential correlation leads to a moving point process with certain spatial Markovian properties.

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1. Introduction

A stochastic flow in a Euclidean space R_d is a family $X = (X_{st}, 0 \leq s \leq t < \infty)$, where $X_{st} = X_{s,t}$ is a random mapping of R_d into (not necessarily onto) itself, satisfying X_{ss} = identity and $X_{tu} \circ X_{st} = X_{su}$, $0 \leq s \leq t \leq u < \infty$, possibly only 'almost surely' in a prescribed sense. $X_{st}(z)$ is the value of X_{st} at $z \in R_d$. Unless otherwise specified our flows are *homogeneous*, both temporally ($s \to s+h$ and $t \to t+h$ for fixed $h > 0$ leaves the law unchanged) and spatially ($X_{st}(z_0+z) - z_0$ and $X_{st}(z)$ have the same law considered as processes in (s, t, z)). A homogeneous flow is called *Brownian* if

(i) $(X_{st}(z), t \geq s)$ is continuous in t;

(ii) if $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots$, then $X_{s_1t_1}, X_{s_2t_2}, \ldots$ are independent.

We do not necessarily have continuity in z.

It is intuitively clear that (ii) is equivalent to:

(ii′) for each $k = 1, 2, \ldots, z_1, \ldots, z_k \in R_d$, and $s \geq 0$ the k-dimensional process

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$(X_{st}(z_1), \ldots, X_{st}(z_k), t \geq s)$ is Markovian. We study only the case, perhaps the only one possible, where these processes are diffusions, which from spatial homogeneity may be assumed to have drift 0. Then from homogeneity, for each z, $(X_{st}(z), t \geq s)$ must be a d-dimensional Brownian motion.

In [16] the author studied Lebesgue-incompressible isotropic flows in R_2 with a smooth covariance structure. In this case the transformations X_{st} are homeomorphisms of R_2 onto itself. Other studies mentioned briefly below have also been concerned with homeomorphic cases, with one exception known to the author: Arratia's work [1] and [2] on flows in R_1 where any two points are independent Brownian before meeting, when they coalesce into a single Brownian point.

In the present paper we study Brownian flows in R_1, the flow of [1] then being a limiting case. We shall be much occupied with coalescence, which occurs only if the covariance structure is not smooth. However the smooth case is of interest too and is used to derive a result about the coalescing case (Theorem 10.5 and Corollary 10.6). Coalescing flows perhaps exist in R_2 and R_3.

A Brownian flow in R_1 is determined by a stationary covariance function b, $b(x) = \lim_{t \downarrow 0} EX_{0t}(0)(X_{0t}(x) - x)/t$. We assume throughout that b satisfies

(1.1) Conditions. b is real continuous nonnegative definite, $b(0) = 1$ (for convenience only), the spectral distribution of b is not of the pure jump type, and b is Lipschitz outside each interval $(-c, c)$, $c > 0$.

Note that the matrix $(b(x_i - x_j))_{i,j=1}^{d}$ is strictly positive definite iff the x_i are all distinct.

We show the existence of a unique Brownian flow in R_1 corresponding to a given b satisfying (1.1). Associated with the flow is a triangular system of stochastic differential equations. For fixed t, the mapping $x \to X_{0t}(x)$ is a point in the space Λ of monotone increasing right continuous mappings of R_1 onto itself. It is shown that the Λ-valued process $t \to X_{0t}$ has a.s. continuous sample functions if Λ is appropriately metrized.

For a flow in R_1 the probability that two distinct points meet is (i) 1 or (ii) 0. From Feller's criterion of accessibility we have (i) or (ii) according as $\int_0^1 x(1 - b(x))^{-1} \, dx$ is $< \infty$ or ∞. If $b''(0)$ is finite (whence $b'(0) = 0$) we have an important subcase (ii') of (ii), where $X_{st}(x)$ has a version continuous in (s, t, x) and X_{st} is a homeomorphism of R_1 onto R_1. If $1 - b(x) \geq c|x|^{2-\varepsilon}$ near 0, for some c, $\varepsilon > 0$, we have an important subcase (i') of (i) where, for each $t > 0$, $X_{0t}(R_1)$ is a countable set with no limit points, a kind of behavior first discovered by Arratia [1] for flows with independent coalescing paths. It is not known whether any other behavior besides that of (i') and (ii') can occur.

In case (i'), the point process $(\beta_i) = (\beta_i^t) = X_{0t}(R_1)$ ($t > 0$ fixed) has a finite intensity ρ_t about which we get some information. The set $\{\beta_i\}$ is the range of the jump-function $x \to X_{0t}(x)$, whose points of jump are another point process $(\alpha_i) = (\alpha_i^t)$, also with intensity ρ_t. As t varies, the process $t \to (\beta_i^t)$ is a (temporally) Markovian process of moving coalescing points.

We show that, in case (ii′), $(X_{0t})^{-1}$ and X_{0t} have the same distribution (t fixed). It is shown from this that, in case (i′), (α_i) and (β_i) have the same distribution, a result shown by different methods for the flow in [1].

The case $b(x)=\mathrm{e}^{-c|x|}$ is given special attention. Letting $X(x)$ be the path $t\to X_{0t}(x)$, $0\leq t<\infty$, we show that then $(X(x), x\in R_1)$ is Markov. Hence the moving system $t\to(\beta_i^t)$ has a spatial Markovian property: conditioning on the path beginning at x_0 makes those beginning on the left independent of those beginning on the right. Further properties of this process are found.

A word about other stochastic flows, considering here, for convenience only, the case $d=1$ with zero drift. Consider an Ito system

$$\mathrm{d}\xi_{st}=\sum_{i=1}^{m}\sigma_i(\xi_{st})\,\mathrm{d}W_{it},\quad t\geq s,$$

where $W_1,\ldots,W_m$ are independent Wiener processes and the σ_i satisfy appropriate conditions. Write the solution as $\xi_{st}=\xi_{st}(x)$ where $\xi_{ss}(x)=x$. Then ξ_{st} is a flow in R_1. From our point of view, note that

$$E(\xi_{0t}(x)-x)(\xi_{0t}(y)-y)=tb(x,y)+\mathrm{o}(t),$$

where $b(x,y)=\sum_{i=1}^{m}\sigma_i(x)\sigma_i(y)$ is spatially inhomogeneous but degenerate, reflecting the continuous dependence of the whole system on a finite number of Wiener processes. Flows of this type are treated in Ikeda and Watanabe [17, Chapter 5], including flows on manifolds. The reader may find additional references there. Here we mention the work of Bismut [7] and Elworthy [11]; the methods of the latter paper can apparently be extended to treat infinite-dimensional driving Wiener processes. The infinite dimensional case has been treated by Baxendale [4, 5], Le Jan [19a], Le Jan and Watanabe [19b], and Kunita [18a].

As far as the author knows, the results of these papers, dealing with noncoalescing cases, do not include any in the present paper except Theorem 8.2; see also the remarks above (13.3).

The present paper makes extensive use of methods or results in [1, 16], and Stroock and Varadhan [22]. Statements or needed modifications of results in these references are given as concisely as possible.

Notation. x, y denote points of R_1, z a point of R_d. The 'measurable' sets in R_d are always the Borel sets $\mathscr{B}_d$. $C_b^m=C_b^m(R_d)$ $[C_0^m]$ are the functions $R_d\to R_1$ which, with their first m derivatives, are bounded and continuous [and have compact support]. $|S|$ is the cardinality of a set S. (W_t) is always Wiener and $W_{1t},\ldots,W_{kt}$ are independent Wiener. If (ξ_t) and (η_t) are processes, 'ξ is adapted to η' means ξ_t depends measurably on η_s, $0\leq s\leq t$. If we write $\mathrm{d}\xi_t=\sigma(\xi_t)\,\mathrm{d}W_t+a(\xi_t)\,\mathrm{d}t$, we make the usual assumptions (see, e.g. [22, Chapter 5]) but it is not assumed that ξ is adapted to W or vice versa. If A is an event depending on a process ξ in R_d, $P_z(A)$ indicates a probability calculated with $\xi_0=z$. c is a constant, not always the same.

2. Preliminaries on diffusion

We use the martingale framework of [22]. Let $\Omega = \Omega_d$ be the space of continuous mappings $\omega : [0, \infty) \to R_d$ with the topology of uniform convergence on compact sets. Putting $\xi_t(\omega) = \omega(t) = \omega_t$, let $\mathcal{M}_t = \mathcal{M}_{dt}$ be the σ-field generated by ξ_s, $0 \leq s \leq t$, $\mathcal{M}_d = \bigvee_t \mathcal{M}_t = \mathcal{M}$.

Define $\mathcal{A}$ operating on C_b^∞ by

$$\mathcal{A}f(z) = \tfrac{1}{2} \sum_{i,j=1}^{d} b_{ij}(z) \frac{\partial^2 f(z)}{\partial x_i\, \partial x_j} + \sum_{i=1}^{d} \beta_i(z) \frac{\partial f(z)}{\partial x_i}, \quad z = (x_1, \ldots, x_d)$$

where $B = (b_{ij})$ is nonnegative definite and B and β are bounded and continuous. A probability measure P on $\mathcal{M}$, governing a process (ξ_t) in R_d, is a solution to the martingale problem (MP) for $\mathcal{A}$ from $z \in R_d$ if $P(\xi_0 = z) = 1$ and if $(f(\xi_t) - \int_0^t \mathcal{A}f(\xi_s)\, ds, \mathcal{M}_t, P)$ is a martingale for each $f \in C_b^\infty(R_d)$.

(2.1) Definitions. $S_t : \Omega_d \to \Omega_d$ is defined by

$$(S_t\omega)(s) = \omega(t+s), \quad s, t \geq 0.$$

If $C \in \mathcal{M}$, if $S_t C \subset C$, if P solves the MP for $\mathcal{A}$, and if $P(C) = 1$, call P a *C-solution.*

We observe that several basic results proved in [22] under the assumption that the MP has a unique solution from each z are also true if we assume only that there is a unique C-solution from each z. In particular:

(2.2) Lemma. *Suppose for each $z \in R_d$ there is a unique C-solution P_z from z. Then*
(a) *if $A \in \mathcal{M}$, $P_z(A)$ is measurable in z;*
(b) *$\{P_z, z \in R_d\}$ is a strong Markov family of probability measures.*

The arguments in [22, 6.7.4 for (a) and 6.2.2 for (b)] are almost unchanged, 'unique C-solution' playing the role of 'unique solution' in [22]. From temporal homogeneity, it is sufficient to consider only the starting time 0.

Henceforth let $B = B_d = (b(x_i - x_j))_{i,j=1}^d$, where b satisfies (1.1). Let D be the set of points in R_d with all coordinates distinct, $H = R_d \backslash D$. As noted in (1.1), $B(z)$ is strictly positive definite if $z \in D$ and degenerate if $z \in H$.

For stochastic differential equations we require a matrix $S = (\sigma_{ij})$ such that $SS^* = B$. A solution to the set of stochastic differential equations

$$dX_i = \sum_{j=1}^{d} \sigma_{ij}\, dW_j, \quad i = 1, 2, \ldots, d,$$

then has the diffusion matrix B. A lower triangular form for S is very useful.

(2.3) Lemma. *Under the condition* (1.1) *there is a matrix* $S = S_d = (\sigma_{ij})$, $1 \leq i, j \leq d$, $\sigma_{ij} = 0$ for $j > i$, *such that* (a) $SS^* = B$; (b) *each σ_{ij} is bounded in R_d, continuous in*

D, and Lipschitz in each compact subset of D. S is uniquely determined by the equations

$$b(x_i - x_j) = \sum_{r=1}^{j} \sigma_{ir}\sigma_{jr}, \quad j = 1, 2, \ldots, i, \; i = 1, 2, \ldots, d, \tag{2.4}$$

if we agree that if $x_i \neq x_k$, $k = 1, 2, \ldots, i-1$ *then* σ_{ii} *(necessarily* $\neq 0$*) is* >0; *while if* $x_i = x_k$ *for some* $k < i$ *we agree that* $\sigma_{ji} = 0$, $j = 1, 2, \ldots, d$. *In particular* $\sigma_{11} = 1$ *and* $\sigma_{j1} = b(x_j - x_1)$. *(Continuity at points of H is not asserted.)*

Proof. The existence of S follows essentially from the Gram–Schmidt orthogonalization procedure. The continuity in D and local Lipschitz properties are readily apparent by induction on i and j, noting that $\sigma_{i1} = b(x_i - x_1)$. Boundedness is clear because $\sum_j \sigma_{ij}^2 = b(x_i - x_i) = 1$.

(2.5) Remark. We see by induction that σ_{ij}, $j \leq i$ is a function of $x_1, x_2, \ldots, x_j$, and x_i and does not depend on d if $i, j \leq d$.

(2.6) Lemma. *Define*

$$\mathscr{A}_d f(z) = \tfrac{1}{2} \sum_{i,j=1}^{d} b(x_i - x_j) \frac{\partial^2 f}{\partial x_i \, \partial x_j}.$$

Let ξ^1 *and* ξ^2 *be solutions to the MP for* $\mathscr{A}_d$ *from* $z \in D$ *with respective hitting times* τ_1 *and* τ_2 *for H,* $\tau_1, \tau_2 \leq \infty$. *Then* $(\tau_1, \xi_s^1, 0 \leq s < \tau_1)$ *and* $(\tau_2, \xi_s^2, 0 \leq s < \tau_2)$ *have the same law.*

The proof, relying on the local Lipschitz character of B_d in D, uses standard arguments and is omitted. (See e.g. of [22, 4.5.2] and Friedman [14, Chapter 5, Section 2].

3. The finite dimensional processes

(3.1) Definitions. Henceforth $C = C_d$ is specialized to be the set of ω in Ω_d such that if $\omega_i(t) = \omega_j(t)$ for some $i \neq j$ and $t \geq 0$, then $\omega_i(t+s) = \omega_j(t+s)$ for all $s \geq 0$. (See Definition 2.1). Define $\mathscr{A}_d f(z)$ as in (2.6). The domain of $\mathscr{A}_d$ will be $C_b(R_d)$ unless otherwise specified.

(3.2) Lemma. *For each* $z \in R_d$ *there is a unique C-solution* P_z *to the MP for* $\mathscr{A} = \mathscr{A}_d$ *from z. If* $A \in \mathscr{M}_d$, $P_z(A)$ *is measurable in z. The family* $\{P_z, z \in R_d\}$ *is strong Markov and Feller.*

Sketch of proof. The only problem is that B_d is degenerate on H (definitions below (2.2)) and not necessarily Lipschitz there. We use induction. If $d = 1$, P_x is clearly Wiener starting at x, measurable in x, and strong Markov. Suppose the result proved for dimensions $\leq d-1$. Fix $z' = (y_1, y_2, \ldots, y_d)$. First suppose $z' \in H$. For specificity suppose that for some $k \leq d-1$ we have $y_j = y_{n(j)}$, $k+1 \leq j \leq d$, $1 \leq n(j) \leq k$, making no assumption as to whether $y_1, \ldots, y_k$ are distinct. Note that $\mathscr{A}_d f(y_1, \ldots, y_d) = \mathscr{A}_k g(y_1, \ldots, y_k)$, where g is defined by $g(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, x_{n(k+1)}, \ldots, x_{n(d)})$, the simplest case being $\mathscr{A}_d f(y_1, y_1, \ldots, y_1) = \frac{1}{2}g''(y_1)$, where $g(x_1) = f(x_1, \ldots, x_1)$. This assertion can be verified by a direct calculation from the definitions of $\mathscr{A}_d$ in (2.6) if $k = d-1$, then using downward induction for smaller values of k. Moreover if f depends only on $x_1, \ldots, x_k$, then, with a little abuse of notation, $\mathscr{A}_d f = \mathscr{A}_k f$. It follows that $(\xi_1(t), \ldots, \xi_k(t), \xi_{n(k+1)}(t), \ldots, \xi_{n(d)}(t))$ is a C-solution for $\mathscr{A}_d$ from z' iff $(\xi_1(t), \ldots, \xi_k(t))$ is one for $\mathscr{A}_k$ from $(y_1, \ldots, y_k)$. Hence we have a C-solution P_z for each $z \in H$. Using the inductive hypothesis it is a unique C-solution. The measurability of $P_z(A)$ for z varying in H is easily established from (2.2a).

Now suppose $z \notin H$. Let P'_z solve the MP for $\mathscr{A}_d$ from z (for existence see [22, 6.1.7]) and let ξ'_t be a corresponding canonical process. Let $\tau = \inf\{t: \xi'_t \in H\}$, $\tau \leq \infty$. Let P_z be the law governing a process ξ having the same law as ξ' until τ; then, on the set $\{\tau < \infty\}$, $(\xi(\tau+s), s \geq 0)$, conditioned by the pre-τ σ-field γ_τ, is given the law $P_{\xi(\tau)}$, which is determined because $\xi(\tau) \in H$. From [22, 6.1.2], which extends to allow $\tau \leq \infty$ if the statement there is modified slightly, P_z solves the MP for $\mathscr{A}_d$ from z. Clearly P_z is a C-solution. Any C-solution ξ'' from z is governed by P_z until reaching H, from (2.6). Using [22, 6.1.3], with the same remark about $\tau \leq \infty$, $(\xi''(\tau+s), s \geq 0)$, conditioned on $\mathscr{F}_\tau$, solves the MP from $\xi''(\tau(\omega))$ a.s. and hence has the law $P_{\xi''(\tau)}$. It follows that ξ'' has the same law as ξ. Measurability follows from (2.2a).

The Feller property will be shown in (3.6). □

(3.3) Consistency lemma. *If $1 \leq k < d$, and if $(\xi_{1t}, \ldots, \xi_{dt})$ is a C-solution of the MP for $\mathscr{A}_d$, then $(\xi_{1t}, \ldots, \xi_{kt})$ is a C-solution for $\mathscr{A}_k$. (Similarly for other subsets.)*

Proof. If $f(x_1, \ldots, x_d) = g(x_1, \ldots, x_k)$, where $g \in C_b^\infty(R_k)$, then $(\mathscr{A}_d f)(x_1, \ldots, x_d) = (\mathscr{A}_k g)(x_1, \ldots, x_k)$. Hence $(\xi_{1t}, \ldots, \xi_{kt})$ solves the MP for $\mathscr{A}_k$; it is clearly the unique C-solution for $\mathscr{A}_k$.

(3.4) Lemma. *Let ξ be a C-solution for $\mathscr{A}_d$. Let $\eta_t = \xi_{it} - \xi_{jt}$, $i \neq j$. Then η is a diffusion in R_1 with absorbing state 0 and operator*

$$\mathscr{G}f(y) = (1 - b(y))f''(y), \quad f \in C_b^\infty.$$

Proof. From known results, or arguing as in (2.2), there is a unique C-solution to the MP for $\mathscr{G}$ from any initial point. Putting $f(x_1, \ldots, x_d) = h(x_i - x_j)$, $h \in C_b^\infty(R_1)$, we see that $f \in C_b^\infty(R_d)$ and $\mathscr{A}_d f = \mathscr{G}h(x_i - x_j)$, whence if $(\xi_{1t}, \ldots, \xi_{dt})$ is the C-solution for $\mathscr{A}_d$, then $\xi_{it} - \xi_{jt}$ is the C-solution for $\mathscr{G}$.

(3.5) Remarks. Since η is a martingale of unchanging sign, $\lim_{t\to\infty}\eta_t$ exists a.s. and must be 0 because $1-b(y)>0$ for $y\neq 0$. From this and Feller's criterion for accessibility, the probability that η reaches 0 is 0 or 1 according as $\int_0^1 y\,\mathrm{d}y/(1-b(y))$ is ∞ or $<\infty$.

(3.6) Corollary to (3.4). *The C-solution* (P_z) *of* (3.2) *has the Feller property.*

Proof. If $(\xi_{1t},\ldots,\xi_{dt},\xi'_{1t},\ldots,\xi'_{dt})$ is the C-solution for $\mathscr{A}_{2d}$ from $(z,z')=(x_1,\ldots,x_d,x'_1,\ldots,x'_d)$ then from (3.3), ξ and ξ' are C-solutions for $\mathscr{A}_d$ from z and z'. From (3.4) and (3.5), if t is fixed $\xi'_{it}-\xi_{it}\to 0$ in law when $x_i\to x'_i$, which implies the Feller property.

For later use we note that η, as regards law, has the representation

$$\eta_t=\eta_0+\int_0^t\sqrt{2(1-b(\eta_s))}\,\mathrm{d}W_s. \tag{3.7}$$

If $b''(0)$ is finite (whence $b'(0)=0$), then using Ito's lemma and arguing as for (5.4) in [16] we find

$$E|\eta_t|^\alpha\leq|\eta_0|^\alpha\,\mathrm{e}^{\alpha(\alpha-1)c_1t},\qquad \alpha\geq 1,\quad c_1=\sup\frac{1-b(y)}{y^2}\leq\tfrac{1}{2}|b''(0)|. \tag{3.8}$$

4. Construction and properties of X_{st}

Since closely related constructions have been given in [1] and [16], conciseness is desirable here. However, some differences in the treatment are required. Theorem 4.7, used in the present construction and believed to be new, may be of independent interest. If $b''(0)$ is finite, the construction is carried out just as in [16].

Let b satisfy (1.1); Ω_1 and $\mathscr{M}_1$ are as in Section 2. Let Γ be the set of functions $\gamma: R_1\to\Omega_1$; let $\mathscr{F}_\Gamma$ be the σ-field in Γ generated by the sets $\{\gamma:\gamma(x)\in B\}$, $x\in R_1$, $B\in\mathscr{M}_1$ or equivalently by the sets $\{\gamma:\gamma_t(x)\leq a\}$, $t\geq 0$, $x\in R_1$, $a\in R_1$. Let $X^*(x,\gamma)=\gamma(x)$. We define a probability measure P on $(\Gamma,\mathscr{F}_\Gamma)$ such that $(X^*(x_1),\ldots,X^*(x_d))$ has the distribution of the C-diffusion determined by $\mathscr{A}_d$ starting at $(x_1,\ldots,x_d)$, $d=1,2,\ldots$; $x_1,\ldots,x_d\in R_1$. The existence and uniqueness of the Ω_1-valued random field $(X^*(x), x\in R_1)$ follow from Lemmas 3.2 and 3.3. The argument is as in [16, Section 2].

Let $X^*_t(x)$ be the value of $X^*(x)$ at t. From the construction, $X^*_t(x,\gamma)$ is continuous in t for each (x,γ). In case $b''(0)$ is finite, we can show as in [16, Section 6], using (3.8) above, that $X^*_t(x)$ has a version $X_t(x)$ continuous in (t,x).

Let Q' be the dyadic numbers in R_1. We adopt the convention that primed letters x', x'', y'', etc. denote points in Q'.

From the properties of the diffusions governed by $\mathscr{A}_d$ and the martingale property of $(X_t^*(x)-X_t^*(y), t\geq 0)$, we see that there is a $\Gamma_0\in\mathscr{F}_\Gamma$, $P(\Gamma_0)=0$ such that $\gamma\notin\Gamma_0$ implies

(a) $X_0^*(x',\gamma)=x', x'\in Q'$;

(b) for each $t\geq 0$, $X_t^*(x',\gamma)$ is nondecreasing in x';

(c) for each x',

$$\lim_{y'\to x'}\sup_{0\leq t<\infty}|X_t^*(y',\gamma)-X_t^*(x',\gamma)|=0;$$

(d) for each x', y' the pair $X^*(x',\gamma)$, $X^*(y',\gamma)$ has the 'C-property' (if they ever become equal they stay equal);

(e) $\lim_{x'\to\pm\infty} X_t^*(x',\gamma)=\pm\infty$.

Regarding (e), since $X_t^*(n)$ is Wiener starting at n, $P\{X_t^*(n)\leq a$ for infinitely many positive integers $n\}=0$, t and a fixed, by the Borel–Cantelli lemma. From the ordered dependence on x', (e) follows for $+\infty$, and similarly for $-\infty$.

Following [1] define X by

$$\begin{aligned} X_t(x,\gamma)&=\inf\{X_t^*(y',\gamma): y'\geq x\} \quad &\gamma\notin\Gamma_0,\\ X_t(x,\gamma)&=x, &\gamma\in\Gamma_0. \end{aligned}\tag{4.1}$$

Then $X_t(x,\gamma)$ is right-continuous nondecreasing in x, and for fixed t, x we have $X_t^*(x)=X_t(x)$ a.s. It is readily seen that (a), (b), and (e) above hold if X^* is replaced by X and x', y' by $x, y\in R_1$. Regarding (c) and (d) see (4.10) and (4.11). We shall see that X is continuous in t. It is easily seen that $X_t(x,\gamma)$ is measurable in (t,x,γ).

(4.2) Definitions. Let Λ be the set of non-decreasing right-continuous functions $\lambda: R_1\to R_1$, $\mathscr{F}_\Lambda$ the σ-field generated by the sets $\{\lambda:\lambda(x)\leq a\}$, $x, a\in R_1$. (Note that $(\Lambda,\mathscr{F}_\Lambda)$ is a standard Borel space.)

For fixed t the mapping from Γ onto Λ given by $\gamma\to X_t(\cdot,\gamma)$ is measurable and hence we have a Λ-valued process $(X_t, t\geq 0)$. X_t is adapted to the σ-field $\mathscr{F}_t^X$ in Γ generated by X_s, $0\leq s\leq t$.

(4.3) Definitions. Let Q_t be the distribution of X_t. If Q' and Q'' are probability distributions of independent elements λ and μ in Λ, let $Q' * Q''$ be the distributions of $\lambda\circ\mu$.

(4.4) Lemma. $Q_t * Q_s=Q_{t+s}$, $s, t\geq 0$.

The proof is essentially the same as that of Lemma 2.8 of [16].

(4.5) Remarks. If follows from the Markovian nature of the finite set processes that $(X_t,\mathscr{F}_t^X)$ is Markov, and from (4.4) that the (stationary) transition function is $p(h,\lambda,A)=P\{X_h\circ\lambda\in A\}$, $\lambda\in\Lambda$, $A\in\mathscr{F}_\Lambda$. Moreover for each $s\geq 0$, $(X_{t+s}, t\geq 0)$ has the same law, up to equivalence, as $(X_t'\circ X_s, t\geq 0)$ where X' is an independent copy of X.

(4.6) Definition. For $n = 1, 2, \ldots$ define a seminorm in the space of locally bounded Borel measurable functions $\lambda : R_1 \to R_1$ by

$$\|\lambda\|_n = \sup_{|x| \leq n} |\lambda(x)|.$$

Define a metric Δ in Λ by

$$\Delta(\lambda, \mu) = \sum_{n=1}^{\infty} 2^{-n} \|\lambda - \mu\|_n (1 + \|\lambda - \mu\|_n)^{-1}.$$

Then (Λ, Δ) is a complete nonseparable metric space.

(4.7) Theorem. *For each $T > 0$, $(X_t, 0 \leq t \leq T)$ is a.s. uniformly continuous in the Δ-metric.*

The proof is based on the following simple lemma, which uses only the ordered nature of the flow and the law of the one-point processes.

(4.8) Lemma. *If $a < b \in R_1$, $t > 0$, $0 < \varepsilon < 1$,*

$$P\{\sup_{\substack{a \leq x \leq b \\ 0 \leq s \leq t}} |X_s(x) - x| > \varepsilon\} \leq \left(\frac{8(b-a)}{\varepsilon} + 2\right) \text{Prob}\{\max_{0 \leq s \leq t} W_s > \tfrac{1}{2}\varepsilon\},$$

where W is a Wiener process.

Proof. Take $a = 0$. We may assume $b \in Q'$. Let X^* denote the original version; x' and s' denote points of Q'. Since $X_t^*(x') = X_t(x')$, $X_t(x)$ is right-continuous in x, and $X_t^*(x)$ is continuous in t, we have

$$\sup_{0 \leq s \leq t} \sup_{0 \leq x \leq b} (X_s(x) - x) = \sup_{0 \leq s \leq t} \sup_{0 \leq x' \leq b} (X_s(x') - x')$$

$$= \sup_{0 \leq x' \leq b} \sup_{0 \leq s' \leq t} (X_{s'}^*(x') - x'),$$

measurability of the sup being now evident. For $j/2^n \leq x' < (j+1)/2^n$, $j = 0, 1, \ldots$, $n = 1, 2, \ldots$, we have

$$X_s^*(x') - x' \leq X_s^*((j+1)2^{-n}) - (j+1)2^{-n} + 2^{-n}.$$

Hence, for $0 < \varepsilon < 1$,

$$P\{\sup_{\substack{0 \leq x \leq b \\ 0 \leq s \leq t}} (X_s(x) - x) > \varepsilon\} \leq P\left\{\sup_{0 \leq s \leq t} \sup_{0 \leq j \leq [2^n b]} \left(X_s^*\left(\frac{j+1}{2^n}\right) - \frac{j+1}{2^n}\right) > \varepsilon - 2^{-n}\right\}$$

$$= P\left\{\sup_j \sup_{0 \leq s \leq t} \left(X_s^*\left(\frac{j+1}{2^n}\right) - \frac{j+1}{2^n}\right) > \varepsilon - 2^{-n}\right\}$$

$$\leq ([2^n b] + 1) \text{Prob}\{\sup_{0 \leq s \leq t} W_s > \varepsilon - 2^{-n}\}.$$

Taking $n = 1+\log_2(1/\varepsilon)+\theta, 0 \leq \theta < 1$, we have

$$P\{\sup_{\substack{0\leq x\leq b\\0\leq s\leq t}}(X_s(x)-x)>\varepsilon\}\leq\left(\frac{4b}{\varepsilon}+1\right)P\left\{\sup_{0\leq s\leq t}W_s>\frac{\varepsilon}{2}\right\}.$$

We complete the proof with a similar argument for the infimum.

Proof of (4.7). Take $T=1$; let $n\geq 1$ and k be integers, $0\leq k\leq 2^n-1$. t' and x' will be in Q', and t and t' will be in $[k2^{-n},(k+1)2^{-n}]$ unless otherwise stated. Using the right-continuity of $X_t(x)$ in x and the continuity of $X_t(x')=X_t^*(x')$ in t and changing the order of suprema where convenient, we find, for $N=1,2,\ldots,$

$$\begin{aligned}U_{knN}&\stackrel{\text{def}}{=}\sup_t\|X_t-X_{k2^{-n}}\|_N=\sup_{|x'|\leq N}\sup_{t'}|X_{t'}(x')-X_{k2^{-n}}(x')|\\&=\sup_{t'}\sup_{|x|\leq N}|X_{t'}(x)-X_{k2^{-n}}(x)|.\end{aligned}$$

Enlarging the probability space, let X' be an independent copy of X. From (4.5) and (4.8)

$$\begin{aligned}P\{U_{knN}>\varepsilon\}&=EP\{\sup_{0\leq t'\leq 2^{-n}}\sup_{|x|\leq N}|X'_{t'}(X_{k2^{-n}}(x))-X_{k2^{-n}}(x)|>\varepsilon|X_{k2^{-n}}\}\\&\leq E\left\{\frac{8(X_{k2^{-n}}(N)-X_{k2^{-n}}(-N))}{\varepsilon}+2\right\}2\operatorname{Prob}(W_{2^{-n}}>\varepsilon/2)\\&=\left(\frac{32N}{\varepsilon}+4\right)\operatorname{Prob}(W_{2^{-n}}>\varepsilon/2).\end{aligned}$$

Hence $P\{\bigcup_{k=0}^{2^n-1}(U_{knN}>\varepsilon)$ for infinitely many $n\}=0$, whence (4.7) follows. □

(4.9) Definition. Take $\Gamma_1\in\mathcal{F}_T$ such that $P(\Gamma_1)=0$, $\Gamma_1\supset\Gamma_0$, and $\gamma\notin\Gamma_1$ implies $X_t(\gamma)$ is uniformly Δ-continuous on finite t-intervals.

From (4.7) and Dini's theorem:

(4.10) Lemma. *If $\gamma\notin\Gamma_1$, then $X_t(x,\gamma)$ is continuous in t, and $X_t(x_n,\gamma)$ converges down to $X_t(x,\gamma)$ uniformly on finite t-intervals whenever $x_n\downarrow x$.*

A result comparable to (4.10) was proved in [1] without using (4.7), required here because the nature of the function $x\to X_t(x)$ has not been settled in all cases (see Section 7).

(4.11) Lemma. *If $\int_0^1 x(1-b(x))^{-1}\,dx=\infty$ (in particular if $b''(0)$ is finite), or if b satisfies (7.3), then almost surely: all pairs $X_\cdot(x_1)$, $X_\cdot(x_2)$ have the C-property (see* (d) *above* (4.1)).

In the former case the result is clear because no pair $X_\bullet(x_1)$, $X_\bullet(x_2)$ with $x_1 \neq x_2$ can meet. In the latter case X_t, for $t > 0$, has a countable discrete range (see (7.4)) and the argument is just as for the comparable case in [1].

Having found a 'nice' version $X_t(x)$, corresponding to $X_{0t}(x)$ of our desired flow X_{st}, we ask if there is an equally nice version of $X_{st}(x)$ (see Section 1). We summarize as follows.

(4.12). *If b satisfies* (1.1) *there is a unique* (*up to equivalence*) *A-valued process X^*_{st}, $0 \leqslant s \leqslant t < \infty$, such that*

(a) *if $s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant \cdots$, $X^*_{s_1t_1}, X^*_{s_2t_2}, \ldots$ are independent*;

(b) *the distribution of X^*_{t-s} is Q_{t-s}* (*see* (4.3));

(c) *for fixed $s \leqslant t \leqslant u$, $X^*_{tu} \circ X^*_{st} = X^*_{su}$ a.s. Also* (Q_t) *determines b uniquely.*

The argument is like that of (2.10) of [16].

(4.13). If $b''(0)$ *is finite, X^*_{st} has a version X_{st} such that $X_{st}(x)$ is continuous in s, t, x; $X_{tu}(X_{su}(x))$ holds with no exceptional s, t, u, or x.* The argument is like that of (6.2) of [16], and uses (3.8).

(4.14). *If b satisfies* (7.3), *then there is a version $X_{st}(x)$ that is continuous in t, nondecreasing in x, and satisfies $X_{tu}(X_{st}(x)) = X_{su}(x)$, $0 \leqslant s \leqslant t \leqslant u < \infty$, $x \in R_1$ with no exceptions.*

Related results in [2] suggest that right-continuity in x may not be possible in (4.14).

The proof of (4.14) is rather lengthy and is omitted, since we shall not need it, and a comparable result has been proved in [2].

5. Motion of adjacent intervals

Let $f(x_0, x_1, x_2) = g(x_1 - x_0, x_2 - x_1)$, where $g \in C^2_b(R_2)$. Then, putting $y_i = x_i - x_{i-1}$, $i = 1, 2$, define $\mathcal{G}_2$ by

$$\mathcal{A}_3 f(x_0, x_1, x_2) = \mathcal{G}_2 g(y_1, y_2)$$

$$= (1 - b(y_1))\frac{\partial^2 g}{\partial y_1^2} + (1 - b(y_2))\frac{\partial^2 g}{\partial y_2^2} - \Gamma(y_1, y_2)\frac{\partial^2 g}{\partial y_1 \partial y_2}, \quad (5.1)$$

$$\Gamma(y_1, y_2) = 1 + b(y_1 + y_2) - b(y_1) - b(y_2). \quad (5.2)$$

The six strict orderings of x_0, x_1, x_2 correspond to six open sets in (y_1, y_2) space determined by the coordinate axes and the line $y_1 = y_2$. The strict positive definiteness of $\mathcal{A}_3$ when x_1, x_2, x_3 are distinct implies the strict positive definiteness of $\mathcal{G}_2$ in each of the six open sets. If (ξ_1, ξ_2, ξ_3) is a C-solution for $\mathcal{A}_3$, then $(\xi_2 - \xi_1, \xi_3 - \xi_2) =_{\text{def}} (\eta_1, \eta_2)$ is a solution for $\mathcal{G}_2$, each component being absorbed at 0. Almost as for (3.4) we see there is only one such solution. Hence (η_1, η_2) is a diffusion. Similarly if $x_0 < x_1 < \cdots < x_n$ and

$$\eta_{it} = X_t(x_i) - X_t(x_{i-1}) \text{ then } (\eta_1, \eta_2, \ldots, \eta_n)$$

is a diffusion. This is not true in general for nonadjacent intervals.

The stochastic equations for (η_1, η_2) in triangular form are

$$\mathrm{d}\eta_1 = [2(1-b(\eta_1))]^{1/2}\,\mathrm{d}W_1, \tag{5.3}$$

$$\mathrm{d}\eta_2 = -\frac{\Gamma(\eta_1, \eta_2)}{[2(1-b(\eta_1)]^{1/2}}\,\mathrm{d}W_1 + [2(1-b(\eta_2))]^{1/2}\left[1 - \frac{\Gamma^2(\eta_1, \eta_2)}{4(1-b(\eta_1))(1-b(\eta_2))}\right]^{1/2}\mathrm{d}W_2; \tag{5.4}$$

the fractions are defined as 0 if η_1 or $\eta_2 = 0$. It is possible to use (5.3) and (5.4) to study the conditional behavior of η_2, given η_1, along the lines of a related procedure given in Section 13.

6. Example related to continuous state branching processes.

Take $b(x) = 1 - |x|$, $|x| \leq 1$, $b(x) = 0$ if $|x| \geq 1$. Let $(\eta_1, \eta_2, \ldots, \eta_n)$ be the lengths of n adjoining intervals, going from left to right. Let $\tau = \inf\{t: \sum \eta_{it} = 1\}$. Since the infinitesimal off-diagonal covariances are 0 for $t < \tau$ we see that $\eta_1, \eta_2, \ldots, \eta_n$ have the same law as n independent continuous state branching processes $\eta_1', \eta_2', \ldots, \eta_n'$ until either sum reaches 1; the law for η_i' is given by $\mathrm{d}\eta_i' = \sqrt{2\eta_i'}\,\mathrm{d}W_i$.

7. The range of $(X_t(R_1))$

Arratia [1] proved that for independent coalescing Brownian paths

$$P\{X_t(R_1) \cap I \text{ is finite} = 1\}, \qquad t > 0, \tag{7.1}$$

whenever I is a compact interval in R_1. Perhaps this holds for Brownian flows whenever points coalesce, i.e. when

$$\int_0^1 \frac{y\,\mathrm{d}y}{1-b(y)} < \infty. \tag{7.2}$$

(See (3.5).) We shall see at least that (7.1) holds provided b satisfies (1.1) and

$$1 - b(x) \geq c|x|^{2-\varepsilon}, \quad |x| \leq c_1, \tag{7.3}$$

for some $\varepsilon > 0$ and $c_1 > 0$. If $1 - b(x)$ is not $O(x^2)$ and (7.3) does not hold, the behavior is left undetermined.

If (7.3) holds, (4.11) and (7.1) together imply that a.s. $(X_t(R_1), t > 0)$ is a countable coalescing system.

(7.4) Theorem. *If b satisfies* (7.3) *and* (1.1), *then* (7.1) *holds.*

Proof[1]**.** Since the argument of [1] does not seem feasible here, we proceed by noting that (7.1) is implied by

$$P_y\{\eta_t > 0\} = O(y), \quad t > 0 \text{ fixed}, \tag{7.5}$$

where η is as in (3.4). To see this, let D_n be the dyadic numbers of order n in $[0,1]$, $D = \bigcup D_n$. Then $X_t(D_n)$ is increasing in n and $|X_t(D_n)| = 1 + \nu_n$, where ν_n is the number of integers $j = 0, 1, \dots, 2^n - 1$ such that $X_t(j/2^n) \neq X_t((j+1)/2^n)$. Then $E|X_t(D_n)| = 1 + 2^n P_{2^{-n}}(\eta_t > 0)$, so that (7.5) implies $E|X_t(D)| < \infty$, implying $E|X_t([0,1])| < \infty$ from right continuity. The theorem follows from this.

To prove (7.5), assuming $0 < \varepsilon < 1$ for convenience, consider the diffusion $\mathrm{d}Z = Z^{1-\varepsilon/2}\,\mathrm{d}W$ on $[0, \infty)$, absorbed at 0. The substitution $V = Z^\varepsilon$ gives $\mathrm{d}V = \varepsilon V^{1/2}\,\mathrm{d}W - \frac{1}{2}\varepsilon(1-\varepsilon)\,\mathrm{d}t$. From Feller [12, 6.1], letting the parameter b of that formula go to 0, or calculating independently we have

$$P_v\{V_t > 0\} = \frac{1}{\Gamma(1/\varepsilon)} \int_0^{2v/(\varepsilon^2 t)} \mathrm{e}^{-u} u^{1/\varepsilon - 1}\,\mathrm{d}u = O(v^{1/\varepsilon}),$$

whence $P_z\{Z_t > 0\} = O(z)$. Let σ be the first time Z reaches 0, let $\varphi(x) = \frac{1}{2}x^{2-\varepsilon}(1 - b(x))^{-1}$, and put $Z'(u) = Z(\tau(u))$, defining τ by $u = \int_0^{\tau(u)} \varphi(Z_s)\,\mathrm{d}s$, $\tau(u) < \sigma$. For convenience suppose $c = \frac{1}{2}$ in (7.3). Then $\varphi(x) \leq 1$ for $0 < x \leq c_1$, so $\tau(u) \geq u$. From Dynkin I, [10, Theorem 10.12] the process $Z'(u)$, $\tau(u) < \sigma$, has the same law as η until η reaches 0. Hence if $y > 0$, noting that $P_y\{\eta \text{ ever reaches } c_1\}$ is y/c_1 if $0 < y < c_1$, η being on the natural scale,

$$\begin{aligned}
P_y(\eta_t > 0) &= P_y(\eta_t > 0, \eta \text{ always} \leq c_1) + O(y) \\
&= P_y(Z'_t > 0, Z' \text{ always} \leq c_1) + O(y) \\
&= P_y(Z'_t > 0, Z \text{ always} \leq c_1) + O(y) \\
&= P_y\left(\int_0^\sigma \varphi(Z_s)\,\mathrm{d}s > t, Z \text{ always} \leq c_1\right) + O(y) \\
&\leq P_y\{\sigma > t, Z \text{ always} \leq c_1\} + O(y) \\
&= P_y(Z_t > 0, Z \text{ always} \leq c_1) + O(y) = O(y). \qquad \square
\end{aligned}$$

(7.6) Definition. Let

$$\rho_t = \lim_{n\to\infty} 2^n P_{2^{-n}}(\eta_t > 0) = \lim E|X_t(D_n)| - 1, \quad t > 0;$$

(7.5) implies $\rho_t < \infty$. Note that $P_y(\eta_t > 0)$ is concave in y [18, Problem 2, Section 4.4].

[1] I am indebted to R. Arratia for helpful discussion on this proof. Arratia obtained stronger results for his case.

(7.7) Definition. If (7.3) holds (implying (7.1)), we speak of the *countable-range case.*

In the countable range case, for fixed $t>0$, X_t is a step function described by an orderly point process $\zeta_t=\zeta=(\alpha_i,\beta_i)$ in R_2, where the α_i are the points of jump of X_t and $\beta_i=X_t(\alpha_i)$; here $(\beta_i)=X_t(R_1)$. From the construction (α_i+x,β_i+x), for fixed $x\in R_1$, has the same law as (α_i,β_i), whence (α_i,β_i) is a 'motion' (see [15, especially (4.3)]), (α_i) and (β_i) are stationary, and (β_i) has the same intensity as (α_i), namely ρ_t $(=E|X_t(D)|-1$ from the proof of (7.4)).

Clearly ρ_t is nonincreasing in t. Since $\rho_t\geq P_y(\eta_t>0)/y$ for $t>0$, $y>0$ (see (7.6)) and $\lim_{t\downarrow 0}P_y(\eta_t>0)=1$, we have $\lim_{t\downarrow 0}\rho_t=\infty$. Since

$$P_y\{\eta_{t+1}>0\}\leq\int_{0+}^{A}P_y\{\eta_1\in\mathrm{d}u\}P_u\{\eta_t>0\}+P_y(\eta_1>A)$$

$$\leq P_A(\eta_t>0)y\rho_1+y/A,$$

we have

$$\rho_{t+1}=\lim_{y\downarrow 0}P_y(\eta_{t+1}>0)/y\leq P_A(\eta_t>0)\rho_1+1/A$$

for arbitrary $A>0$. Since $\lim_{t\to\infty}P_A(\eta_t>0)=0$, we have $\lim_{t\to\infty}\rho_t=0$.

Finally, using a result of Feller, we find that $\int_0^1\rho_t\,\mathrm{d}t$ *is finite iff* $\int_0^1(1-b(y))^{-1}\,\mathrm{d}y$ *is finite (i.e. iff* 0 *is regular for* η.) To see this, put $p'(y,s)=\int_0^\infty \mathrm{e}^{-st}P_y(\eta_t=0)\,\mathrm{d}t$. Then, for $s>0$,

$$\int_0^\infty \mathrm{e}^{-st}\rho_t\,\mathrm{d}t=\lim_{y\downarrow 0}\int_0^\infty \mathrm{e}^{-st}\left(\frac{1-P_y(\eta_t=0)}{y}\right)\mathrm{d}t$$

$$=\lim_{y\downarrow 0}\frac{(1/s-p'(y,s))}{y}=-\frac{\partial^+}{\partial y}p'(y,s)\bigg|_{y=0}.$$

From [13, p. 473–474], $\partial p'/\partial y$ remains bounded as $y\downarrow 0$ iff $\int_0^1(1-b(y))^{-1}\,\mathrm{d}y<\infty$, and the above assertion follows.

See (10.6) for more information about (α_i) and (β_i).

8. The case $b''(0)$ finite

Let $Q_t[b]$ be the distribution of X_t constructed from a given b with $b''(0)$ finite. From (4.14), we may suppose X_t is a random point of C_{11}, the continuous functions $R_1\to R_1$ with the compact open topology.

(8.1) Lemma. *Let* K *be a family of real covariances* b *satisfying* (1.1), $\sup_{b\in K}|b''(0)|<\infty$. *Then* $\{Q_t[b], b\in K\}$ *is tight for fixed* t.

The proof follows that of 11.3 of [16], using (3.8) above.

If $b''(0)$ is finite it is clear that the continuous version $X_{st}(x)$ is a homeomorphism, the situation being much simpler than in [16].

It has been shown under various assumptions that flows in Euclidean spaces (or on manifolds) are diffeomorphisms; see, e.g., Bismut [7], Baxendale [4], Elworthy [11]. For completeness we give here the simple proof which handles our case. With a little more work, one can show that $\mathrm{d}X_t(x)/\mathrm{d}x$ has a version continuous in t and x, having for fixed x the exponential-Wiener nature suggested by (8.2).

(8.2.) Theorem. *If $b^{(4)}(0)$ is finite, X_t (t fixed) is a.s. differentiable in x with a differentiable inverse. For fixed t and x the law of $\mathrm{d}X_t(x)/\mathrm{d}x$ is that of* $\exp[(-b''(0))^{1/2}W_t+\frac{1}{2}b''(0)t]$.

Proof of (8.2). Let $\eta_{1t}=X_t(h)-X_t(0)$, $\eta_{2t}=X_t(0)-X_t(-h)$, $h>0$, $\zeta_t=\eta_{1t}-\eta_{2t}$. Using the triangular system

$$\mathrm{d}X_t(h)=\mathrm{d}W_{1t},\qquad \mathrm{d}X_t(0)=\sigma_{21}\,\mathrm{d}W_{1t}+\sigma_{22}\,\mathrm{d}W_{2t},$$

$$\mathrm{d}X_t(-h)=\sigma_{31}\,\mathrm{d}W_{1t}+\sigma_{32}\,\mathrm{d}W_{2t}+\sigma_{33}\,\mathrm{d}W_{3t}$$

(see (2.3)), we find

$$\mathrm{d}\zeta_t=(1-2\sigma_{21}+\sigma_{31})\,\mathrm{d}W_{1t}+(-2\sigma_{22}+\sigma_{32})\,\mathrm{d}W_{2t}+\sigma_{33}\,\mathrm{d}W_{3t}.$$

Using Ito's lemma, taking expectations and simplifying by (2.4) we get, since $\zeta_0=0$,

$$E\zeta_t^n=\tfrac{1}{2}n(n-1)\int_0^t E\{[4(1-b(\eta_{1s}))$$

$$+4(1-b(\eta_{2s}))-2(1-b(\eta_{1s}+\eta_{2s}))]\zeta_s^{n-2}\}\,\mathrm{d}s,\quad n=2,3,\ldots.$$

Since $1-b(x)=cx^2+O(x^4)$ (for convenience take $c=1$) we have, for $n=2$,

$$E\zeta_t^2=2\int_0^t E\{\zeta_s^2+O(\eta_{1s}^4+\eta_{2s}^4)\}\,\mathrm{d}s.$$

But $E\eta_{is}^4\leq ch^4$ from (3.8), keeping t on a compact interval. Hence, from Gronwall's inequality we have $E\zeta_t^2\leq ct\,\mathrm{e}^{2t}h^4$. It follows from known results (see Cramér and Leadbetter [9], Corollary to the theorem of 4.3, where h may be kept bounded) that $(X_t(x), x\in R_1)$ has a continuously differentiable version, t fixed.

Putting $\gamma_t^h=\eta_{1t}/h$, $h>0$, we have, from (3.7),

$$\gamma_t^h=1+\int_0^t\left(\frac{2(1-b(h\gamma_s^h))}{h^2(\gamma_s^h)^2}\right)^{1/2}\gamma_s^h\,\mathrm{d}W_s,$$

Letting $h\downarrow 0$ and using [22, (11.1.4) along with (5.3.1) and (4.5.1)], we find that the law of the process $(\gamma_t^h, t\geq 0)$ converge to that of the solution of $\gamma_t=1+\sqrt{2}\int_0^t\gamma_s\,\mathrm{d}W_s$, i.e. $\gamma_t=\exp(\sqrt{2}\,W_t-t)$, which implies the second part of the theorem, after rescaling. □

9. Stirrings and their limits

If X_{st} plays the role of a Wiener process among composition semigroups, certain 'stirring processes' are analogous to jump-type additive processes. See [16] and [19] for more on stirring processes.

Let

$$\mu \in C_0^3(R_1), \qquad \mu^\alpha(x) = \mu(x-\alpha), \quad x, \alpha \in R_1.$$

Define $F_\tau(x)$, τ, $x \in R_1$ by

$$\frac{\mathrm{d}F_\tau(x)}{\mathrm{d}\tau} = \mu(F_\tau(x)), \qquad F_0(x) = x \tag{9.1}$$

and put $F_\tau^\alpha(x) = \alpha + F_\tau(x-a)$. Note that F_τ^α solves (9.1) with μ replaced by μ^α. Define G_τ and G_τ^α similarly with $-\mu(x)$ and $-\mu(x-\alpha)$ replacing $\mu(x)$ and $\mu(x-\alpha)$; then G_τ^α and F_τ^α are inverse homeomorphisms of R_1 onto R_1.

Let $\{(t_i, \alpha_i)\}$ and $\{(t_i', \alpha_i')\}$ be independent Poisson point processes in $[0, \infty) \times R_1$, each with intensity $n/2$. We define X_{st}^n, $n = 1, 2, \ldots$ as the composition of all the (infinitely many) transformations $F_{1/\sqrt{n}}^{\alpha_i}$ and $G_{1/\sqrt{n}}^{\alpha_i'}$ such that $s < t_i \leq t$, $s < t_i' \leq t$, any two transformations being applied in the same temporal order as their respective t_i's or t_i's. (For details see the more complicated situation in [16].) We call X^n a *stirring process*. Much as in [16] we see that for fixed $s < t$, X_{st}^n converges in law to X_{st}, where X is the Brownian flow for which $b(x) = \int_{-\infty}^{\infty} \mu(\alpha)\mu(\alpha + x)\,\mathrm{d}\alpha$. (We use two families of transformations F and G, rather than one as in [16], to avoid centering problems.) The next result will be used below. Convergence in law refers to weak convergence.

(9.2) Lemma. *Let b satisfy* (1.1) *with $b''(0)$ finite. Then there exists a sequence of stirring processes X^n such that for fixed $s < t$, X_{st}^n converges in law to X_{st}, where X is the Brownian flow corresponding to b.*

The proof is omitted because of its similarity to that of (11.3) and (16.6) of [16]. As in [16], we proceed in stages, using the fact that a limit of limits of stirrings is itself a limit of stirrings. In (9.2) we consider X_{st} ($s < t$ fixed) as a random point in C_{11} (see Section 8).

10. Inverses and duality

In [16] it was shown that if X_{st} is an incompressible Brownian flow in R_2 with an inverse, then $(X_{st})^{-1}$, if it exists, has the same law as X_{st}, $s < t$ fixed. Since incompressibility was used unnecessarily in the proof, a brief proof is given here for the present case. The result applies to a variety of processes which can be approximated by stirrings, and has also been shown by Baxendale for a number of cases.

(10.1) Lemma. *If $b''(0)$ is finite, then, for fixed $s<t$, $(X_{st}(-x), x\in R_1)$ has the same law as $(-X_{st}(x), x\in R_1)$ and X_{st}^{-1} has the same law as X_{st}.*

Proof. Let X^n be a stirring process. Fix $T>0$. From Section 9 we see that $(X_T^n)^{-1}$, which can be constructed by applying the transformations $(F_{1/\sqrt{n}}^{\alpha_i'})^{-1}=G_{1/\sqrt{n}}^{\alpha_i'}$ and $(G_{1/\sqrt{n}}^{\alpha_i'})^{-1}=F_{1/\sqrt{n}}^{\alpha_i'}$ in reverse order of time, has the same law as X_T^n. The second part of the lemma then follows from (9.2). If $x_1,\ldots,x_k\in R_1$, then $(X_t(-x_1),\ldots,X_t(-x_k), t\geq 0)$ and $(-X_t(x_1),\ldots,-X_t(x_k))$ are C-solutions for $\mathscr{A}_k$ with initial point $(-x_1,\ldots,-x_k)$, and hence have the same law, from (3.2). The first part of the lemma follows. □

(10.2) Corollary to (10.1). *If A and B are countable unions of closed subsets of R_1, then*

$$P(X_{st}(A)\cap B\neq\emptyset)=P(X_{st}(B)\cap A\neq\emptyset). \tag{10.3}$$

(See [16, Section 18] for measurability of $X_{st}(A)$ and further comments.) A related but different notion of duality is discussed in [1].

Evidently (10.3) does not hold in the countable-range case since if A is an interval and B is a single point, the left side is 0 and the right side is >0 when $s<t$. Let us see what does remain.

(10.4) Lemma. *Let b satisfy (1.1). Let $b_N(x)=k_N\int_{R_1} b(x-y)Ng(Ny)\,\mathrm{d}y$, where g is the standard normal density and k_N makes $b_N(0)=1$. Let X, X^N be the corresponding Brownian flows. Then the d-point diffusions of X^N converge weakly to those of X, $d=1,2,\ldots$.*

Proof. The matrix $B_d=(b(x_i-x_j))$ is not necessarily Lipschitz, so the uniqueness condition usual for such results is lacking.

Let $\xi_t^N=(\xi_{1t}^N,\ldots,\xi_{dt}^N)=(X_t^N(x_1),\ldots,X_t^N(x_d))$. Consider a compact t-interval, say $[0,1]$. The family $(\xi_t^N, 0\leq t\leq 1)$, $N=1,2,\ldots$ is relatively compact since each component is Wiener. Each weak limit ξ of a subsequence of (ξ^N) solves the MP for $\mathscr{A}_d$, so we need only show ξ is a C-solution (see (3.2)). Let $\eta_t=\xi_{it}-\xi_{jt}=\xi_{2t}-\xi_{1t}$, say, with $\eta_0=y\geq 0$, and $\eta_t^N=\xi_{2t}^N-\xi_{1t}^N$, going now to a convergent subsequence. Take $\eta_0^N=x_2-x_1=y$. Let A be the event that there are random times $0\leq\tau_1<\tau_2\leq 1$ with $\eta(\tau_1)=0$, $\eta(\tau_2)>0$. Let A_{mn} be the event

$$\{\exists s_1,s_2\colon 0\leq s_1<s_2\leq 1,\ \eta(s_1)<1/m,\ \eta(s_2)>1/n\},$$

$$n=1,2,\ldots;\ m=n+1,n+2,\ldots.$$

Let A_{mn}^N be the corresponding event for η^N. Since η^N is a nonnegative martingale and a strong Markov process, $P(A_{mn}^N)\leq n/m$. Now $A_{mn}\downarrow$ in m and $\uparrow$ in n and $A=\bigcup_{n\geq 1}\bigcap_{m\geq n+1}A_{mn}$. Then, for each positive integer k, $P(A)\leq \limsup_{n\to\infty}P(A_{2kn,n})$. Now $A_{2kn,n}$ and $A_{2kn,n}^N$ correspond to the *same open subset* of $C[0,1]$, so from the theory of weak convergence $P(A_{2kn,n}) \leq \liminf_{N\to\infty} P(A^N_{2kn,n}) \leq 1/(2k)$. Hence $P(A)=0$. □

(I am indebted to a referee for improving the original proof.)

(10.5) Theorem. *Let X be the flow corresponding to b satisfying* (1.1). *Let* $X'_t(y) = \sup\{x: X_t(x) \leq y\}$. *Then X'_t, t fixed, considered as a random point of A (Definition* (4.2)), *has the same distribution as X_t.*

(10.6) Corollary. *The point processes (α_i) and (β_i) of Section 7 have the same law.*

Remark. The corollary was discovered and proved by Arratia [1] for independent coalescing Brownian paths, using different methods.

Proof. Fix $t>0$. Since $X_t(x)\to\infty(-\infty)$ as $x\to\infty(-\infty)$, $X'_t(x)$ is defined for each x. Since X_t is nondecreasing and right-continuous in x, so is X'_t. Hence it suffices to show that the finite-dimensional distributions are the same.

From the definition of X'_t

$$(X_t(x) > y) \subset (X'_t(y) \leq x) \subset (X_t(x) \geq y). \tag{10.7}$$

Let $x_1,\ldots,x_k$, $y_1,\ldots,y_k \in R_1$. The multidimensional distribution functions of $(X_t(y_1),\ldots,X_t(y_k))$ and of $(X_t(x_1),\ldots,X_t(x_k))$ are continuous in each variable, because $X_t(x)$ is normal, and hence, as is known, are continuous. Let X^N be the flow of Lemma 10.4. Using (10.7), (10.4) and (10.1), we have

$$\begin{aligned}
&P\{X'_t(y_i) \leq x_i,\ 1\leq i\leq k\} \leq P\{X_t(x_i) \geq y_i,\ 1\leq i\leq k\}\\
&\quad= \lim_{N\to\infty} P\{X^N_t(x_i) \geq y_i,\ 1\leq i\leq k\} = \lim_{N\to\infty} P\{(X^N_t)^{-1}(x_i) \geq y_i,\ 1\leq i\leq k\}\\
&\quad= \lim_{N\to\infty} P\{X^N_t(y_i) \leq x_i,\ 1\leq i\leq k\} = P\{X_t(y_i) \leq x_i,\ 1\leq i\leq k\}.
\end{aligned}$$

Similarly we find

$$P\{X'_t(y_i) \leq x_i,\ 1\leq i\leq k\} \geq P\{X_t(y_i) \leq x_i,\ 1\leq i\leq k\},$$

completing the proof of (10.5). The corollary is true because the points of jump and range of X_t are, respectively the range and points of jump of X'_t. □

Think of $\alpha_{i+1} - \alpha_i$ as the amount of mass mapped by X_t into β_i (see Section 7). Then (10.6) implies that under the respective Palm distributions the stationary sequence of masses $(\alpha_{i+1}-\alpha_i)$ has the same law as the stationary sequence of spacings $(\beta_{i+1}-\beta_i)$.

11. The generating field

As $s\downarrow 0$ the finite dimensional distributions of the field U_s, where $U_s(x) = s^{-1/2}(X_s(x)-x)$, converge to those of the Gaussian field $(U(x), x\in R_1)$ with mean 0 and covariance function b. If $b''(0)$ is finite we also have weak convergence. This suggests calling U the *generating field of* X.

We may define (because we can write down the appropriate covariances) a Gaussian family of fields U_{st}, $0\leq s\leq t<\infty$ with independent (s,t) increments, such that $U_{st}+U_{tu}=U_{su}$ a.s. and U_{st} has the covariance function $(t-s)b$. The relation seems similar to that between certain operator-valued multiplicative semigroups and their additive counterparts, as studied by Bucan [8]. Baxendale [4] used a process comparable to U_{st} in establishing the existence of flows.

12. The Markov case

If $b(x)=\mathrm{e}^{-c|x|}$, the generating field is Markovian. Here is a more general spatial Markov property. The definition of Ω_1 is in Section 2. We take $c=1$.

(12.1) Theorem. *Assume* $b(x)=\mathrm{e}^{-|x|}$. *Let* $(X(x), x\in R_1)$ *be the* Ω_1*-valued process with values* $X(x)=X_\cdot(x)$, *the path from x. Then X is Markovian. The same is true if* $X(x)=(X_t(x), 0\leq t\leq T)$ *where* $T<\infty$ *is fixed.*

Proof. Given $x_1,\ldots,x_{d+1}$, determine σ_{ij}, $1\leq i,j\leq d+1$, as in Section 2, taking $b(x)=\mathrm{e}^{-|x|}$. Let $W'_1,\ldots,W'_{d+1}$ be independent standard normal random variables and define $U_i=\sum_{j=1}^{i}\sigma_{ij}W'_j$. Then $U_1,\ldots,U_{d+1}$ are distributed jointly as $U(x_1),\ldots,U(x_{d+1})$, where U is the field in Section 11. Note $U_1=W'_1$ and $\sigma_{i1}=b(x_i-x_1)$. Then $E(U_{d+1}|W'_1,\ldots,W'_d)=\sum_{j=1}^{d}\sigma_{d+1,j}W'_j$.

Suppose $x_d\leq x_{d-1}\leq\cdots\leq x_1\leq x_{d+1}$. Assuming first that the inequalities are strict so that $(W'_1,\ldots,W'_d)$ is a function of $(U_1,\ldots,U_d)$ (see Section 2), we have $E(U_{d+1}|W'_1,\ldots,W'_d)=E(U_{d+1}|U_1,\ldots,U_d)$, and this, from the Gaussian and Markov properties of U, is $b(x_{d+1}-x_1)U_1=b(x_{d+1}-x_1)W'_1$. Comparing with the above, we see that $\sigma_{d+1,2}=\sigma_{d+1,3}=\cdots=\sigma_{d+1,d}=0$. The argument is easily modified if some of the x_i coincide. Since $EU^2_{d+1}=1$, $\sigma_{d+1,d+1}=(1-b^2(x_{d+1}-x_1))^{1/2}$.

Let $\xi=(\xi_1,\ldots,\xi_d)$ be a C-solution to the MP for $\mathscr{A}_d$ from $(x^0_1,\ldots,x^0_d)$, $x^0_d\leq x^0_{d-1}\leq\cdots\leq x^0_1$. From [22, (4.5.2)] we can suppose ξ satisfies in law (see (2.5))

$$\xi_{it}=x^0_i+\int_0^t\sum_{j=1}^{i}\sigma_{ij}(\xi_{1s},\ldots,\xi_{js},\xi_{is})\,\mathrm{d}W_{js},\quad 1\leq i\leq d,\tag{12.2}$$

where we do not assert that $(\xi_1,\ldots,\xi_d)$ is 'strong', i.e. adapted to $(W_1,\ldots,W_d)$, the usual assumption justifying that assertion being lacking; but note $\xi_{1t}=x^0_1+W_{1t}$. Adjoin to (12.2) the equation

$$V_t=x^0_{d+1}+\int_0^t b(V_s-\xi_{1s})\,\mathrm{d}W_{1s}+\int_0^t\sqrt{1-b^2(V_s-\xi_{1s})}\,\mathrm{d}W'_s,\tag{12.3}$$

where (a) $W'(=W_{d+1})$ is independent of $\xi_1, \dots, \xi_d$, $W_1, \dots, W_d$; (b) V is adapted to (ξ_1, W'); (c) if $\tau = \inf\{t\colon V_t = \xi_{1t}\}$, then $\tau < \infty$ a.s. and $V_t = \xi_{1t}$ for all $t \geq \tau$ a.s. We shall see in (12.5) that this can be done. Then $\xi_{dt} \leq \cdots \leq \xi_{1t} \leq V_t$ a.s. Noting the values of the $\sigma_{d+1,j}$ determined above because of the exponential form of b, we see that V satisfies

$$V_t = x_{d+1}^0 + \int_0^t \sum_{j=1}^{d+1} \sigma_{d+1,j}(\xi_{1s}, \dots, \xi_{ds}, V_s)\, \mathrm{d}W_{js}. \tag{12.4}$$

From (3.2) and [22, 5.3.1], $(\xi_1, \dots, \xi_d, V)$ is the C-solution to the MP for $\mathscr{A}_{d+1}$ from $(x_1^0, \dots, x_{d+1}^0)$. Since V is adapted to (W', W_1) and W' is independent of $\xi_1, \dots, \xi_d$, $W_1, \dots, W_d$, we see that conditioning V on $\xi_1, \dots, \xi_d$ gives the same result as conditioning on ξ_1, which proves the result. □

(12.5) Lemma. (12.3) *has a solution* V *with properties* (a)–(c) *above.*

Proof. If $x_{d+1}^0 = x_1^0$, $V = \xi_1$. Assuming $x_{d+1}^0 > x_1^0$, enlarge the probability space and let W' be a Wiener process as in (a). Let $b_n(x) = b(x) = \mathrm{e}^{-|x|}$ for $|x| \geq 1/n$, $b_n(x) = \mathrm{e}^{-nx^2}$ for $|x| \leq 1/n$, and consider the equation

$$V_t^n = x_{d+1}^0 + \int_0^t b_n(V_s^n - \xi_{1s})\, \mathrm{d}W_{1s} + \int_0^t \sqrt{1 - b_n^2(V_s^n - \xi_{1s})}\, \mathrm{d}W'_s. \tag{12.6}$$

For fixed n the functions $x \to b_n(u-x)$ and $x \to \sqrt{1 - b_n^2(u-x)}$ are bounded, with Lipschitz constant independent of u. The usual method of successive approximations yields the unique solution of (12.6), adapted to (W_1, W'). Since (ξ_{1t}, V_t^n) is a diffusion with diffusion matrix

$$\begin{pmatrix} 1 & b_n(v-x) \\ b_n(v-x) & 1 \end{pmatrix},$$

then $\eta_t^n = V_t^n - \xi_{1t}$ is a diffusion with 0 drift and diffusion coefficient $2(1 - b_n(y))$. Hence η^n is a martingale, which never reaches 0 if $x_{d+1}^0 - x_1^0 > 0$, but necessarily $\lim \eta_t^n = 0$ a.s.

Let $\tau_n = \inf\{t\colon \eta_t^n = 1/n\}$, $\tau = \sup \tau_n$. Standard considerations show that $m < n$ implies $V_t^m = V_t^n$ for $t \leq \tau_m$. Put $V_t = \lim V_t^n$ for $t < \tau$, $V_t = \xi_{1t}$ for $t \geq \tau$. Then V is a.s. continuous. Comparing V with the diffusion on $[0, \infty)$ having 0 drift and diffusion coefficient $2(1 - b(y))$, we see that $\tau < \infty$ a.s. Since $V_t = 1_{(\tau > t)} \lim V_t^n + 1_{(\tau \leq t)} \xi_{1t}$, V is adapted to (W_1, W').

If $\varepsilon > 0$, $P\{\sup_{t > \tau_{n^2}} \eta_t^{n^2} > \varepsilon\} \leq 1/(n^2 \varepsilon)$, whence $\sup_{t \geq \tau} |V_t^{n^2} - V_t| = \sup_{t \geq \tau} |V_t^{n^2} - \xi_{1t}| \leq \sup_{t > \tau_{n^2}} \eta_t^{n^2} \to 0$ a.s. as $n \to \infty$. Next,

$$\sup_{\tau_{n^2} < t < \tau} |V_t^{n^2} - V_t| \leq \sup_{t > \tau_{n^2}} \eta_t^{n^2} + \sup_{\tau_{n^2} < t < \tau} |\xi_{1t} - V_t| \to 0 \quad \text{a.s.}$$

from the preceding and the fact that $\tau_{n^2} \uparrow \tau$. Since $V_t^{n^2} = V_t$ for $t \leq \tau_{n^2}$, a.s. $\lim_{n \to \infty} V_t^{n^2} = V_t$ uniformly in t. From [14, 4.2.7] we get the desired result. □

13. Conditional law of a trajectory

Given $x_1 < x_2$, let $\xi_t = X_t(x_1)$, $\eta_t = X_t(x_2) - X_t(x_1)$. We study the law of η conditioned on ξ, or equivalently the law of $X_\bullet(x_2)$ conditioned on ξ. Using (3.2), we see that (ξ, η) is, in law, the unique solution to the MP for $\mathscr{A}_2'$:

$$\mathscr{A}_2' f = \tfrac{1}{2} f''_{xx} + (1-b(y)) f''_{yy} - (1-b(y)) f''_{xy}$$

such that η is absorbed at 0. From [22, 4.5.2] we can take

$$d\xi_t = d W_{1t}, \tag{13.1}$$

$$d\eta_t = -(1-b(\eta_t))\, d W_{1t} + \sqrt{1-b^2(\eta_t)}\, d W_{2t}. \tag{13.2}$$

Arguing much as in the proof of (12.5), we can arrange that η is adapted to (W_1, W_2). The system (13.1)–(13.2) has a very simple structure. Nevertheless standard filtering formulas e.g. in [20, I, Chapter 8] do not seem very helpful. Our method is to make a transformation getting rid of W_1 in (13.2). We shall work out the details only in the Markov case $b(x) = e^{-|x|}$. Some (although less!) information has been obtained whenever $1 - b(x) \sim x$ as $x \downarrow 0$, with some additional smoothness requirements on b in $(0, \infty)$. Possibly the method used here is a special case of that in the general results on filtering in Bismut and Michel [6], which the author learned about recently.

Until further notice we assume $\xi_0 = 0$. Consider the differential equation

$$\frac{dy}{dx} = -(1-b(y)), \qquad y(0) = y_0 \in R_1. \tag{13.3}$$

For the moment assume only that b satisfies (1.1) and is Lipschitz in R_1. Then the solution never changes sign. Suppose we find $\varphi(x, y)$ with continuous partial derivative of order ≤ 2 such that for each $y_0 \geq 0$ (note this restriction) the graph of $\varphi(x, y) = \varphi(0, y_0)$ is a solution curve of (13.3), so that $\partial\varphi/\partial x - (1-b(y))\partial\varphi/\partial y = 0$ if $y \geq 0$. Putting $V_t = \varphi(\xi_t, \eta_t)$ in (13.2), before conditioning, we get, since η remains ≥ 0 (φ and its derivatives are evaluated at (ξ_t, η_t), b at η_t)

$$d V_t = \varphi_y'(1-b^2)^{1/2}\, d W_{2t} + \tfrac{1}{2}[\varphi''_{xx} - 2\varphi''_{xy}(1-b) + 2\varphi''_{yy}(1-b)]\, dt. \tag{13.4}$$

We may now condition on ξ (i.e. on W_1) by taking ξ to be a nonrandom continuous function in (13.4). Specializing now to $b(x) = e^{-|x|}$, we take $\varphi(x, y) = e^x(e^y - 1)$. Then (13.4) becomes

$$d V_t = \sqrt{2 V_t(\exp(\xi_t) + \tfrac{1}{2} V_t)}\, d W_{2t} + \tfrac{1}{2} V_t\, dt, \quad t \geq 0, \ V_0 \geq 0. \tag{13.5}$$

V is now adapted to W_2 if ξ is fixed, since ξ and η are adapted to W_1 and (W_1, W_2) respectively. We may think of (13.5) as providing a transition law from a continuous function ξ to a continuous function V.

Here is an application of (13.5). The usefulness of (13.5) is that it shows V to be rather like a branching process when it is small.

(13.6) Theorem. *Let* $b(x)=\mathrm{e}^{-|x|}$. *If* $x_1-h<x_1<x_2<x_2+h$ *and* $t>0$,

$$P\{X_t(x_1)-X_t(x_1-h)>0,\ X_t(x_2+h)-X_t(x_2)>0\}\leqslant c_t h^2,$$

where $c_t<\infty$ *does not depend on* x_1 *and* x_2.

(13.7) Corollary. *Let* $\alpha=(\alpha_i)$ *and* $\beta=(\beta_i)$ *be the point processes of Section 7 for this* b. *Then*

$$E|\alpha\cap[0,1]|^2=E|\beta\cap[0,1]|^2<\infty.$$

Proof. Treating ξ as non random define $\tau(t)$ by

$$t=\int_0^{\tau(t)}(\mathrm{e}^{\xi(s)}+\tfrac{1}{2}V_s)\,\mathrm{d}s. \tag{13.8}$$

Because of the recurrence of Brownian motion we can assume $\int_0^\infty \mathrm{e}^{\xi(s)}\,\mathrm{d}s=\infty$, so that τ is uniquely determined for each $t\geqslant 0$. Let $\phi(u)=\mathrm{e}^{-1}\times$ Lebesgue measure $\{s\colon s\leqslant u,\ \xi_s\geqslant -1\}$, $\phi^{-1}(t)=\inf\{u\colon \phi(u)=t\}$. Then $\phi\phi^{-1}(u)=u$, $\phi^{-1}\phi(u)\leqslant u$, $t\geqslant \phi(\tau_t)$, $\tau^{-1}(u)\geqslant\phi(u)$, and $\tau_t\leqslant\phi^{-1}(t)$ since $\int_0^u(\mathrm{e}^{\xi}+\frac{1}{2}V)\,\mathrm{d}s\geqslant\phi(u)$.

Let $V'(t)=V(\tau(t))$. Using Section 2.8 of McKean [21], we transform (13.5) to

$$\mathrm{d}V_t'=\sqrt{2V_t'}\,\mathrm{d}W_t'+\tfrac{1}{2}V_t'(\mathrm{e}^{\xi(\tau_t)}+\tfrac{1}{2}V_t')^{-1}\,\mathrm{d}t. \tag{13.9}$$

Here W' is an appropriate Wiener process and V_t', W_t' and $\xi(\tau_t)$ are now $\mathcal{F}_{\tau_t}$-measurable; $(\mathcal{F}_t)$ is the family of σ-fields generated by W_2.

The coefficient of $\mathrm{d}t$ is $\leqslant\frac{1}{2}V_t'\exp[-m(\tau_t)]\leqslant\frac{1}{2}V_t'\exp[-m(\phi^{-1}(t))]$, where $m(u)=\inf\{\xi_s\colon 0\leqslant s\leqslant u\}$. We compare (13.9) with

$$\mathrm{d}V_t''=\sqrt{2V_t''}\,\mathrm{d}W_t'+\tfrac{1}{2}V_t''K\,\mathrm{d}t,\quad 0\leqslant t\leqslant T, \tag{13.10}$$

where $K=K(\xi,T)>\exp[-m(\phi^{-1}(T))]$, and V'' is the solution of (13.10) absorbed at 0 with $V_0''=V_0'>0$. As in the proof of (12.5) we can make V'' adapted to W'. We should expect the solution (3.10) to be stochastically larger than that of (13.9). Comparison results known to the author (e.g. Yamada and Ogura [23]) do not quite fit the present case. However, making the transformations $V^*=\sqrt{2V'}$, $V^{**}=\sqrt{2V''}$, modified so as to be smooth in a small neighborhood of 0, we obtain, not too close to 0,

$$\mathrm{d}V^*=\mathrm{d}W'+\tfrac{1}{2}(\tfrac{1}{2}V^*g^*-1/V^*)\,\mathrm{d}t,$$

$$\mathrm{d}V^{**}=\mathrm{d}W'+\tfrac{1}{2}(\tfrac{1}{2}V^{**}K-1/V^{**})\,\mathrm{d}t,$$

where g^* is adapted to $\mathcal{F}_{\tau(t)}$ and $0<g^*<K$. From this, as others have observed in similar cases, we see that $V_t^{**}\geqslant V_t^*$ in law. Then, using P' to denote conditional

probability for fixed W_1 (see (13.1)),

$$\begin{aligned} P'\{V'_t > 0 \mid V'_0 = v\} &\le P'\{V''_t > 0 \mid V''_0 = v\} \\ &= 1 - \exp\{-\tfrac{1}{2} vK\, e^{Kt/2}(e^{Kt/2}-1)^{-1}\} \\ &\le \left(\tfrac{1}{2}K + \frac{1}{t}\right) v. \end{aligned} \tag{13.11}$$

(See Athreya and Ney [3, V1.6.15], in which we supply a factor $(-\frac{1}{2})$ in the exponent. Hence

$$\begin{aligned} P'\{V_u > 0 \mid V_0 = v\} &= P'\{V'(\tau^{-1}(u)) > 0 \mid V'_0 = v\} \\ &\le P'\{V'(\phi(u)) > 0 \mid V'_0 = v\} \\ &\le v\left\{\tfrac{1}{2}\exp[-m(\phi^{-1}\phi(u))] + \frac{1}{\phi(u)}\right]\right\} \\ &\le v\left\{\tfrac{1}{2}\exp(-m(u)) + \frac{1}{\phi(u)}\right\}. \end{aligned} \tag{13.12}$$

In terms of the original system (13.1) and (13.2), (13.12) says, for fixed ξ, and $u > 0$,

$$P'\{\eta_u > 0 \mid \eta_0 = y\} \le (e^y - 1)\left\{\tfrac{1}{2}\exp\left(-m(u) + \frac{1}{\phi(u)}\right)\right\}. \tag{13.13}$$

It is easily verified that if ξ is taken to be Wiener, $\exp(-m(u))$ and $1/\phi(u)$ have finite second moments, $u > 0$.

Now consider two intervals $(-a-\Delta, -a)$ and $(0, \Delta)$, $a \ge 0$, $\Delta > 0$. Let ξ be the path from 0, ξ' the path from $-a$. From the Markov property, if ξ' and ξ are given, the behavior of a trajectory to the right of ξ is independent of one to the left of ξ'. Hence, from (13.13),

$$\begin{aligned} &P\{X_t(-a-\Delta) \ne X_t(-a), X_t(0) \ne X_t(\Delta)\} \\ &\quad \le E(e^{\Delta}-1)^2\left(\tfrac{1}{2}e^{-m(t)} + \frac{1}{\phi(t)}\right)\left(\tfrac{1}{2}e^{-m'(t)} + \frac{1}{\phi'(t)}\right) \end{aligned}$$

where m' and ϕ' refer to $-\xi' - a$ as m and ϕ do to ξ. From the second moment properties above we have

$$P\{X_t(-a-\Delta) \ne X_t(-a), X_t(0) \ne X_t(\Delta)\} \le c\Delta^2, \quad a \ge 0,\ 0 \le \Delta \le 1, \tag{13.14}$$

where c depends only on t. □

Fix t. Let $I_{jn} = 1$ if $X_t(j/2^n) \ne X_t((j+1)/2^n)$, $0 \le j \le 2^n - 1$, and let $\nu_n = \sum_{j=0}^{2^n-1} I_{jn}$. From (7.5) and (13.14), $E(\nu_n)^2 \le c$, where c is different from above, but independent of n. The proof of the corollary is completed by referring to (10.6), (7.4), and the discussion below (7.7).

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